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IDENTITIES WITH INVOLUTION FOR THE MATRIX ALGEBRA OF ORDER TWO IN CHARACTERISTIC p

BY

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ABSTRACT

Let $M_2(K)$ be the matrix algebra of order two over an infinite field K of characteristic $p \neq 2$. If K is algebraically closed then, up to isomorphism, there are two involutions of first kind on $M_2(K)$, namely the transpose and the symplectic. If K is not algebraically closed, studying $*$-identities it is still sufficient to consider only these two involutions. We describe bases of the polynomial identities with involution in each of these cases.

Introduction

Suppose R is an algebra with involution $*$ over an infinite field K, and denote by $T(R,*)$ the ideal of all $*$-polynomial identities in R. The description of $T(R,*)$ is an important task in PI theory. Quite a lot of research has been done in the direction of finding the identities with involution of minimal degree satisfied by the given algebra. We refer to two recent papers, [3, 4], and their bibliography for further information on the topic. Another important question in this direction consists in describing the generators of $T(R,*)$ as an ideal of $*$-identities. The positive results on this last question are sparse, the only nontrivial cases being those of $R = M_2(K)$, the matrix algebra of order 2 over a field K when K is either of characteristic 0 [13], or K is finite [14]. Also when $R = G$, the infinite dimensional Grassmann algebra over a field of characteristic 0, the problem was solved in [1].

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In this paper we describe bases of the $*$-identities for the algebra $M_2(K)$ when K is an infinite field, $\operatorname{char} K = p \neq 2$. We consider both the transpose and the symplectic case. It should be noted that every involution of the first kind on the full matrix algebra $M_n(K)$ is equivalent either to the transpose or to the symplectic, the latter being possible only when n is even. We make use of the description of the weak polynomial identities for the pair $(M_2(K), sl_2(K))$ where $sl_2(K)$ is the Lie algebra of the 2×2 traceless matrices, see [10]. While this is sufficient to deal with the case of the symplectic involution, in order to proceed with the transpose we need invariant theory of the classical groups [5] and the description of the identities of $M_2(K)$ given in [12].

1. Preliminaries

In this paper we consider algebras, vector spaces, modules etc., over a fixed infinite field K, $\operatorname{char} K \neq 2$. Unless otherwise stated, all algebras we consider will be associative and unitary. Let R be an algebra with centre $Z(R)$, and suppose $*$ is an involution on R. Denote $Z(R,*) = \{r \in Z(R) \mid r^* = r\}$. The involution $*$ is of first kind if $Z(R) = Z(R,*)$, otherwise $*$ is of second kind (see for example [16, Def. 2.2.16, p. 121]). Note that this definition is not the usual one but an equivalent form of it. All necessary facts about algebras with involution and their identities can be found in Rowen's book [16], Chapters 2, 3, 7.3, 7.4. We recall some of them later on.

Let $T = \{t_i \mid i \in \mathbb{N}\}$ and $S = \{s_i \mid i \in \mathbb{N}\}$ be two disjoint countable sets of variables; we form the free unitary associative algebra $K\langle T, S\rangle$ freely generated over K by the union of these two sets. One defines an involution $*$ on this algebra setting $t_i^* = s_i$ and $s_i^* = t_i$ for all i. Then $K\langle T, S, *\rangle$ is the free algebra with involution. We adopt the notation T^* and t_i^*, respectively, for S and for s_i. If $f = f(t_1, \ldots, t_n, t_1^*, \ldots, t_n^*) \in K\langle T, T^*\rangle$ then f is an identity with involution, or else an $*$-identity, if $f(r_1, \ldots, r_n, r_1^*, \ldots, r_n^*) = 0$ for every choice of $r_i \in R$. Ideals of $*$-identities are defined in the natural way in analogy with the case of ordinary polynomial identities. Every such ideal I is closed under the $*$-endomorphisms of $K\langle T, T^*\rangle$, and I is the ideal of $*$-identities of an algebra with involution, for example of the quotient algebra $K\langle T, T^*\rangle / I$. If f, g are two polynomials in $K\langle T, T^*\rangle$ then g is a consequence of f if g lies in the least ideal of $*$-identities containing f. If in addition f is a consequence of g then f and g are equivalent as $*$-identities. If I is ideal of $*$-identities then the set $B \subseteq I$ is a basis of I when I coincides with the ideal of $*$-identities generated by B.

It is known that if $*$ is an involution of the second kind on a matrix algebra then every $*$-identity is an ordinary identity as well, see [16, Proposition 2.3.39, p. 132]. Hence in the case of $M_2(K)$, $\operatorname{char} K \neq 2$, we consider involutions of the first kind only since we know a basis of the ordinary polynomial identities of $M_2(K)$, see [12]. Furthermore, in [12] minimal bases of the identities of $M_2(K)$ were described.

From now on we consider only involutions of the first kind. It is well known that if R is central simple algebra over K then there are at most two equivalence classes of involutions (of first kind) on R. We refer to [16, Chapter 3.1, pp. 167–171] for the definition of the equivalence of involutions and related topics.

We shall specialize the above to the matrix algebras $M_n(K)$. The following two involutions on $M_n(K)$ are well known. The transpose involution is $a \mapsto a^t$ where a^t is the usual matrix transpose for $a \in M_n(K)$. The symplectic involution is $a \mapsto a^s$ for $n = 2m$. Here if $a = \begin{pmatrix} b & c \\ d & e \end{pmatrix} \in M_n(K)$ is a block matrix where b, c, d, $e \in M_m(K)$ then

$$a^s = \begin{pmatrix} e^t & -c^t \\ -d^t & b^t \end{pmatrix}.$$

According to [16, Corollary 3.1.58] the following description of the involutions on $M_n(K)$ holds. If n is odd, then every involution $*$ on $M_n(K)$ is equivalent to the transpose. If n is even, then every involution $*$ is equivalent either to the transpose or to the symplectic. Furthermore, there is an extension field F of K such that $M_n(K) \otimes_K F \equiv M_n(F)$ with the involution induced by $*$ is **isomorphic** to $M_n(F)$ with either the transpose or the symplectic involution. In particular, since the field K is infinite, the ideal of identities of $(M_n(K), *)$ coincides either with that of $(M_n(K), t)$ or with that of $(M_n(K), s)$. See [16, Theorem 3.1.61, pp. 169–170].

One may write $x_i = t_i + t_i^*$ and $y_i = t_i - t_i^*$; then since $\operatorname{char} K \neq 2$, one gets $t_i = (1/2)(x_i + y_i)$ and $t_i^* = (1/2)(x_i - x_i^*)$, that is, x_i is the symmetric component of r_i and y_i is the skew symmetric one. Furthermore, one may take $X = \{x_i \mid i \in \mathbb{N}\}$ and $Y = \{y_i \mid i \in \mathbb{N}\}$ as the free generators of $K\langle T, S\rangle \cong K\langle X, Y\rangle$. We shall use letters x, with or without lower indices, to denote the variables in X, that is the ones $*$ leaves fixed, and y (again with or without indices) for the elements of Y; for them $*$ changes the sign.

The base field is infinite hence the ideals of $*$-identities of $K\langle X, Y\rangle$ are multihomogeneous with respect to the natural (multi-)grading on $K\langle X, Y\rangle$. Therefore every such ideal is generated by its multihomogeneous elements, and we restrict our attention to such elements.

Let $L(X)$ be the free Lie algebra freely generated over K by X; then $K\langle X\rangle$ is the universal enveloping algebra of $L(X)$. Let $x_1, x_2, \ldots, u_1, u_2, \ldots$ be an ordered basis of the vector space $L(X)$ where all elements are multihomogeneous, the variables x_i are the least in the order and $\deg u_i \geq 2$ for all i. The (associative) subalgebra $B(X)$ of $K\langle X\rangle$ generated by 1 and by all u_i is spanned by products of Lie elements; the elements of $B(X)$ are called **proper** (or commutator) polynomials. It is well known that the T-ideal of an algebra is generated by its proper polynomials. One modifies this to the case of $*$-polynomials in the following way. Since $1 \in K\langle X, Y\rangle$ is symmetric, every proper polynomial is a linear combination of products of skew symmetric variables y followed by a product of commutators. This is a standard argument; see for example [7, Chapter 4.3] for its detailed proof for ordinary identities, and [6, Section 2] for the case of identities with involution. Here the commutators are of the type $[z_1, z_2, \ldots, z_k]$, $k \geq 2$, where z_i may be either x_t or y_s. We assume that the commutators are left normed, that is $[a, b] = ab - ba$ and $[a, b, c] = [[a, b], c]$. Note that all symmetric variables appear in commutators only. We call the polynomials of the type described above, **$*$-proper** polynomials. Thus the $*$-proper multihomogeneous identities determine all $*$-identities of an algebra with involution.

Let $a \circ b = (1/2)(ab + ba)$ be the Jordan product of a and b; then in every associative algebra

$$[a \circ b, c] = a \circ [b, c] + b \circ [a, c]; \quad [c, a, b] = 4c \circ (a \circ b) - 4a \circ (b \circ c).$$

We shall need the notion of **weak polynomial identity**. Let L be a Lie algebra that is a subalgebra of the Lie algebra R^- of an associative algebra R. Here R^- is the vector space R equipped with the Lie product $[a, b] = ab - ba$. Assume that the set L generates R as an algebra. The polynomial $f(x_1, \ldots, x_n) \in K\langle X\rangle$ is a **weak** polynomial identity for the pair (R, L) if $f(a_1, \ldots, a_n) = 0$ in R for every $a_i \in L$. The set $T(R, L)$ of all weak identities for (R, L) is an ideal of $K\langle X\rangle$ that is closed under Lie substitutions. In other words, if $f \in T(R, L)$ then $f(g_1, \ldots, g_n) \in T(R, L)$ for every $g_i \in L(X)$. The weak identities for the pair $(M_2(K), sl_2(K))$ were described by Razmyslov when $\operatorname{char} K = 0$, see for example [15, Theorem 6.5, pp. 56–57], and by the second author when $\operatorname{char} K > 2$, see [10]. Namely, they follow from the weak identity $[a \circ b, c] = 0$. For further details concerning weak identities see [10, 11, 15].

2. The transpose involution

In this section we describe a basis of the identities with the transpose involution $*$ for the algebra $M_2(K)$. First we exhibit several (quite a lot of) $*$-identities and thus reduce significantly the set of polynomials that one has to consider. Then we employ invariant theory and weak identities in order to show that what is "left" is just necessary.

The following polynomials are $*$-identities for $M_2(K)$.

$$[y_1y_2, x], \quad [y_1, y_2], \tag{1}$$

$$[x_1, x_2][x_3, x_4] - [x_1, x_3][x_2, x_4] + [x_1, x_4][x_2, x_3], \tag{2}$$

$$[y_1x_1y_2, x_2] - y_1y_2[x_1, x_2]. \tag{3}$$

Recall that one may substitute the x_i's by symmetric matrices and the y_i's by skew symmetric ones. Hence the validity of the $*$-identities (1) follows from the fact that two skew symmetric 2×2 matrices commute and that their product is a scalar matrix. Analogously $[x_i, x_j]$ evaluated on symmetric matrices produces skew symmetric one, hence $[x_1, x_2][x_3, x_4] = [x_3, x_4][x_1, x_2]$ and (2) is a variation of the standard identity s_4 that holds for the algebra $M_2(K)$. One checks easily that (3) is also $*$-identity for the 2×2 matrices.

We denote by I the ideal of $*$-identities in $K\langle X, Y\rangle$ generated by the identities above; let $R = K\langle X, Y\rangle/I$ be the corresponding relatively free algebra.

We shall deduce some $*$-identities for R that will be used later on. Most of these can be found in [13] in the case of characteristic 0. We give hints for their deduction since the paper [13] is available only in Russian. Since $x_i^* = x_i$ and $y_i^* = -y_i$ then one checks immediately that in $K\langle X, Y\rangle$ the following equalities hold:

$$[x_1, x_2]^* = -[x_1, x_2]; \quad [x, y]^* = [x, y];$$
$$(x_1 \circ x_2)^* = x_1 \circ x_2; \quad (x \circ y)^* = -x \circ y; \quad (y_1 \circ y_2)^* = y_1 \circ y_2. \tag{4}$$

Therefore $[x \circ y_1, y_2] = 0$ in R, and $y_1 \circ [x, y_2] = 0$ as well. Thus $y_1xy_2 = y_2xy_1$ in R, and since $y_1yy_2 = y_2yy_1$ we have that $y_1zy_2 = y_2zy_2$ for $z = x + y$.

From the last identity it follows that in R we have

$$[y_1, x, y_2] = [y_1, x]y_2 - y_2[y_1, x] = 2[y_1, x]y_2, \tag{5}$$

$$[y_1, x_1, y_2, x_2] = 2[[y_1, x_1]y_2, x_2] =$$
$$2y_1y_2[x_2, x_1] + 2[y_1x_1y_2, x_2] = 4y_1y_2[x_2, x_1]. \tag{6}$$

But $[y_1, x_1, y_2, x_2] = [[y_1, x_1], [y_2, x_2]] + [y_1, x_1, x_2, y_2]$ by the Jacobi identity. Now the element $[y_1, x_1, x_2]$ is skew-symmetric in R, hence the second commutator vanishes and we get the identity

$$[[y_1, x_1], [y_2, x_2]] = 4y_1y_2[x_1, x_2]. \tag{7}$$

Similarly we obtain

$$[[y_1, x_1], [y_2, x_2]] = 2[[y_1, x_1], y_2, x_2] = 2[y_1, x_1][y_2, x_2] + 2[y_1, x_1, x_2]y_2,$$

and

$$[y_1, x_1][y_2, x_2] = 2y_1y_2[x_1, x_2] - [y_1, x_1, x_2]y_2. \tag{8}$$

The following lemma was proved in [13, Lemma 1] for multilinear polynomials. The same proof holds for multihomogeneous ones.

LEMMA 2.1: *Let $f \in R$ be a multihomogeneous $*$-proper polynomial. Then $f = f_1 + f_2 + f_3$ where f_1 is a linear combination of products $u_1u_2 \cdots u_{2k}$, f_2 respectively of products $u_1u_2 \cdots u_{2m+1}$, and f_3 of products $u_1u_2 \cdots u_n w$. Here all u_i are skew symmetric commutators of degree ≥ 1 and w is symmetric of degree ≥ 2. Furthermore, every commutator of degree ≥ 2 contains at most one skew symmetric variable.*

Proof: We give a brief idea of the proof in order to keep the exposition self-contained.

STEP 1: By the identity (8) one represents the product of two symmetric commutators, each of degree ≥ 2, as a sum of products of skew symmetric ones.

STEP 2: By the identity $y_1 \circ [x, y_2] = 0$ we obtain that skew symmetric commutators anticommute with the symmetric ones (the latter of degree ≥ 2). Also, the skew symmetric commutators commute according to (1).

STEP 3: If we are given a product of several symmetric and skew symmetric commutators, then using Step 2 we can reorder, up to a sign, the commutators in such a way that the skew symmetric ones precede the symmetric commutators.

STEP 4: Applying several times the procedure from Step 1 to the result of Step 3, we will be left with at most one symmetric commutator, and so we have the representation $f = f_1 + f_2 + f_3$.

Now let the commutator u contain two skew symmetric variables. If $\deg u = 3$ then either all variables are skew symmetric and $u = 0$ according to (1) or

$u = [y_1, x, y_2]$. In the latter case one applies (5). Hence suppose $\deg u \geq 4$, $u = [w, y_1, z, \ldots]$ where $z = y_2$ or $z = x$, and $\deg w \geq 2$. If w is skew symmetric then $u = 0$ by (1), so suppose w is symmetric. But then $w = [v, w_1]$ where v is skew symmetric commutator and w_1 is symmetric, $\deg w_1 \geq 1$. Then we apply (5) and get $[w, y_1] = [v, w_1, y_1] = 2[v, w_1]y_1$. When $z = y_2$ we have

$$[w, y_2, y_2] = 2[v, w_1, y_2]y_1 = 4y_1y_2[v, w_1]$$

and when $z = x$, by (6) we get

$$[w, y_1, x] = [v, w_1, y_1, x] = 4vy_1[x, w_1].$$

Since vy_1 is central element then $u = 4y_1v_1u_1$ for $v_1 = v$ or $v_1 = y_2$ and some commutator u_1. So we may continue by induction on $\deg u$.

The only case left is when $u = [y_1, \ldots, y_2]$ where y_1 and y_2 are the only skew symmetric variables in u, and in it we may suppose that $u = [v, x, y_2]$ for some skew symmetric commutator v. We deal with it using (5) and thus we are done. ∎

Let

$$a_r = \begin{pmatrix} \theta_{11}^{(r)} & \theta_{12}^{(r)} \\ \theta_{21}^{(r)} & \theta_{22}^{(r)} \end{pmatrix}, \quad r = 1, 2, \ldots$$

be generic 2×2 matrices and denote Gen_2 the subalgebra that they generate in $M_2(K[\theta_{ij}^{(r)}])$. Then Gen_2 is the relatively free algebra in the variety of algebras defined by $M_2(K)$. An analogous construction yields $(Gen_2, *)$, the generic algebra with (the transpose) involution, see [6, Section 3]. Let s_r and t_r be generic symmetric and skew symmetric matrices, respectively:

$$s_r = \begin{pmatrix} \chi_{11}^{(r)} & \chi_{12}^{(r)} \\ \chi_{12}^{(r)} & \chi_{21}^{(r)} \end{pmatrix}; \quad t_r = \begin{pmatrix} 0 & \xi^{(r)} \\ -\xi^{(r)} & 0 \end{pmatrix}.$$

A natural way to visualise $(Gen_2, *)$ is to consider $\chi_{ij}^{(r)} = \frac{1}{2}(\theta_{ij}^{(r)} + \theta_{ji}^{(r)})$ and $\xi^{(r)} = \frac{1}{2}(\theta_{12}^{(r)} - \theta_{21}^{(r)})$; then $s_r = (1/2)(a_r + a_r^*)$ and $t_r = (1/2)(a_r - a_r^*)$. According to [6], $(Gen_2, *)$ is isomorphic to the relatively free algebra in the variety of algebras with involution determined by $M_2(K)$ with the transpose involution.

COROLLARY 2.2: *Let $f = f_1 + f_2 + f_3$ be as in Lemma 2.1. Then:*

(i) (Cf. [13, Remark 3] for the multilinear case.) *The polynomial f is $*$-identity for $M_2(K)$ if and only if f_1, f_2, f_3 are.*

(ii) *If f is $*$-identity for $M_2(K)$ then every f_i follows from some f_i', $\deg f_i' \leq \deg f_i$, and f_i' depends on at most one skew symmetric variable, $i = 1, 2, 3$.*

Proof: The proof of (i) is analogous to that of [13]; it is based on the following fact. The polynomial f_1 when evaluated on $M_2(K)$ produces scalar matrices only while f_2 yields skew symmetric ones and f_3 gives symmetric traceless matrices only. But every matrix decomposes uniquely as a sum of three matrices of the above types and thus (i) is done. As to (ii), one observes that the identity

$$[v, x_1, \dots, x_{2k}][y_1, x'_1, \dots, x'_q] = y_1[v, x_1, \dots, x_{2k}, x'_1, \dots, x'_q] \tag{9}$$

holds in R; here x_i, x'_i are symmetric variables and v is a skew symmetric commutator, $\deg v \geq 2$. This identity can be deduced by an easy induction on q. On the other hand, by $y_1 z y_2 = y_2 z y_1$, $z = x + y$, we have $[y_1, x_1, y_2] = [y_2, x_1, y_1] = 2[y_1, x_1]y_2 = 2[y_2, x_1]y_1$ and we can choose the skew symmetric variable that is to be "thrown out" of the commutator. Hence if $f_i = f_i(x_1, \dots, x_a, y_1, \dots, y_b)$ we may assume that y_1 appears in commutators only and no y_i, $i > 1$ is inside a commutator. Then substitute x_i for s_i and y_i for t_i, and write down the matrix in $(Gen_2, *)$ thus obtained as $PQ = 0$ where P equals the product of (powers of) t_i, $1 \leq i \leq b$. This product has nonzero determinant and hence P is generically invertible. So we may cancel out P and we get $Q = 0$. ∎

LEMMA 2.3: *Let f_2 and f_3 be the polynomials from Corollary 2.2 (ii), depending on exactly one skew symmetric variable y. Suppose further that f_2 and f_3 are linear combinations of commutators of degree ≥ 4. Then in R we have*

$$f_2(y, x_1, \dots, x_n) = \alpha[y, x_{i_1}, \dots, x_{i_m}] + 4y g_1(x_1, \dots, x_m);$$
$$f_3(y, x_1, \dots, x_m) = \beta[y, x_{i_1}, \dots, x_{i_m}] + 4y g_3(x_1, \dots, x_m)$$

for suitable α, $\beta \in K$, $i_1 \leq \cdots \leq i_m$ and for some polynomials g_1, g_3 of the types f_1 and f_3 of Lemma 2.1.

Proof: First we consider the case of f_2. If v is a commutator then $[v, x_2, x_1] = [v, x_1, x_2] + [v, [x_2, x_1]]$. Hence if v is skew symmetric, by Identity (1) we get $[v, x_1, x_2] = [v, x_2, x_1]$, and by Identity (6) we have

$$\begin{aligned} [v, x_1, x_3, x_2, x_4] &= [v, x_1, x_2, x_3, x_4] + [v, x_1, [x_3, x_2], x_4] \\ &= [v, x_1, x_2, x_3, x_4] + 4v[x_3, x_2][x_4, x_1]. \end{aligned} \tag{10}$$

Therefore, for $u = [v, x_3, x_4, x_1, x_2]$, using (10) we obtain

$$\begin{aligned} u &= [v, x_3, x_1, x_4, x_2] + 4v[x_4, x_1][x_2, x_3] \\ &= [v, x_1, x_3, x_2, x_4] + 4v[x_4, x_1][x_2, x_3] \\ &= [v, x_1, x_3, x_2, x_4] + 4v[x_3, x_2][x_4, x_1] + 4v[x_4, x_1][x_2, x_3]. \end{aligned}$$

Finally, according to (6), we get

$$[v, x_3, x_4, x_1, x_2] = [v, x_1, x_2, x_3, x_4]. \tag{11}$$

But the last identity means that we may commute the consecutive pairs of variables inside a commutator $[v, \dots]$ where v is skew symmetric. This implies that we may order the variables in a skew symmetric commutator in ascending way, and that the assertion of the Lemma for polynomials f_2 is proved.

The proof for f_3 is rather similar. Let v be skew symmetric commutator, one uses $[v, x_1, x_3, x_2] = [v, x_1, x_2, x_3] + [v, x_1, [x_3, x_2]]$ and $[y_1, x_1, \dots, x_n, y_2] = [y_2, x_1, \dots, x_n, y_1]$. (The latter identity follows easily from $y_1 z y_2 = y_2 z y_1$.) Using the identity (5) one obtains

$$\begin{aligned}[v, x_1, x_3, x_2] &= [v, x_1, x_2, x_3] + 2[[x_3, x_2], x_1]v \\ &= [v, x_1, x_2, x_3] + 2[x_1, x_2, x_3]v.\end{aligned}$$

In this way we may reorder the variables in the case of f_3 as well, and we finish this case in the same manner as that of f_2. ∎

PROPOSITION 2.4: *Let, in the notation of Corollary 2.2, f_1, f_2, f_3 be $*$-identities for $M_2(K)$. Suppose that every f_i contains at most one entry that is a skew symmetric variable. Then, in R, each one of f_1, f_2, f_3 follows from some $*$-identity that does not depend on skew symmetric variables. Assuming that the polynomials f_i do not depend on such variables and that $\deg f_i \geq 4$, we have the following.*

1. *The polynomial f_1 is a linear combination of products of two skew symmetric commutators. One of these two commutators is of degree two.*
2. *The polynomial f_2 is a linear combination of skew symmetric commutators.*
3. *The polynomial f_3 is a linear combination of symmetric commutators.*

Proof: We already proved that each of f_i is a linear combination of products of commutators, and that the only skew symmetric variable (if any) may be placed at the first position of the first commutator in the product.

Consider first f_1. Using the identity (6) we write it as a combination of products of two skew symmetric commutators. If f_1 depends on symmetric variables only, then by (9) we obtain that one of the commutators in each product is of degree 2. If, on the contrary, f_1 depends on one skew symmetric variable y, then by (9) we write it as $f_1(y, x_1, \dots, x_n) = y f_2(x_1, \dots, x_n)$. Therefore f_1 follows from some $*$-identity of the type f_2 which does not depend on skew symmetric variables.

Now consider a polynomial of the type f_2. By (6) it is a linear combination of skew symmetric commutators. If f_2 does not contain any skew symmetric variable we are done. If it does depend on y then by Lemma 2.3 we write

$$f_2(y, x_1, x_2, \ldots, x_n) = \alpha[y, x_{i_1}, x_{i_2}, \ldots, x_{i_m}] + 4y f_1(x_1, x_2, \ldots, x_n)$$

for some polynomial of the type f_1 and $\alpha \in K$. Substitute then

$$y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad x_1 = x_2 = \cdots = x_n = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then $[y, x_{i_1}, x_{i_2}, \ldots, x_{i_m}]$ goes to $\pm 2^{m/2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $2^a \neq 0$ in K. (Note that m must be even.) Since f_1, as a sum of commutators, vanishes under this substitution we have $\alpha = 0$. Now by the generic invertibility of y we get that $f_1(x_1, x_2, \ldots, x_n)$ is $*$-identity in R.

Finally for $*$-identities of type f_3, applying (5) on the symmetric commutator and one of the skew symmetric commutators, we get that f_3 is a linear combination of symmetric commutators. By Lemma 2.3 we reduce this case to that of f_2 without skew symmetric variables, and thus we are done. ∎

The following Corollary summarizes the results above.

COROLLARY 2.5: *Let $f \in R$. If f is an identity for $(M_2(K), *)$ then f follows from some $*$-identity that depends on symmetric variables only.*

Therefore, when $*$ is the transpose involution on $M_2(K)$ one may consider polynomials depending on symmetric variables. Until the end of this section all polynomials we consider will depend only on symmetric variables.

Definition 2.6: 1. Denote by $F_1 \subseteq R$ the span of all products of two skew symmetric commutators where one of the commutators is of degree 2.

2. Denote by $F_2 \subseteq R$ the span of all skew symmetric commutators of degree ≥ 4.

3. Denote by $F_3 \subseteq R$ the span of all symmetric commutators of degree ≥ 4.

In [2, Theorem 5] it was proved that the ordinary identities of $M_2(K)$ when K is infinite, $\operatorname{char} K = p \neq 2$, follow from the identities

$$s_4 = \sum_{\sigma \in S_4} (-1)^\sigma x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} x_{\sigma(4)}, \quad h_5 = [[x_1, x_2] \circ [x_3, x_4], x_5],$$

and, in the case $p = 3$, from one more identity,

$$r_6 = [x_1, x_2] \circ (u \circ v) - (1/8)([x_1, u, v, x_2] + [x_1, v, u, x_2] - [x_2, u, x_1, v] - [x_2, v, x_1, u])$$

where $u = [x_3, x_4]$ and $v = [x_5, x_6]$.

Clearly h_5 follows from the $*$-identities (1). As to r_6 observe first that

$$[x_1, u, v, x_2] = -[u, x_1, x_2, v] = -4uv[x_2, x_1] = [x_1, v, u, x_2]$$

according to (6). On the other hand, $[x_2, u, x_1]$ is skew symmetric; hence the last two commutators in r_6 vanish on R. Thus r_6 annihilates on R.

The above motivates the definition of a linear operator $L(a, b)$ for a, b being symmetric variables, on the direct sum $F_1 \oplus F_2 \oplus F_3$. An analogous operator was first employed by Iltyakov in [9], and subsequently used in [17] and in [12].

Definition 2.7: Let w_1, w_2 be symmetric variables or commutators and let a, b be symmetric variables; define $[w_1, w_2]L(a, b)$ to be

$$(1/8)([w_1, a, b, w_2] + [w_1, b, a, w_2] - [w_2, a, w_1, b] - [w_2, b, w_1, a]).$$

If w_1 and w_2 are skew symmetric commutators and $\deg w_2 = 2$ then we define $(w_1 w_2)L(a, b) = w_1(w_2 L(a, b))$.

It is easy to verify that for the symmetric variables x_1, x_2, a, $b \in R$ one has

$$[x_1, x_2]L(a, b) = (1/4)[x_1, x_2, a, b].$$

Note that $L(a, b) = L(b, a)$.

The following Lie identities are satisfied in the Lie algebra $sl_2(K)$:

$$\begin{aligned} [x_1, x_2, x_3]L(a, b) &= [[x_1, x_2]L(a, b), x_3], \\ [x_1, a]L(x_2, b) - [x_2, a]L(x_1, b) &= (1/4)[x_1, x_2, b, a], \\ [x_1, x_2]L(a, b)L(c, d) &= [x_1, x_2]L(c, d)L(a, b); \end{aligned}$$

see [17, Identities (3), (4), (5)]. All of them (as well as all the identities for $sl_2(K)$ when $\operatorname{char} K \neq 2$) follow from one identity of degree 5; see the main theorem of [17]. It is well known that all Lie identities of the matrix algebra $M_2(K)$ follow from the standard polynomial s_4 of degree 4; see for example [8]. But

$$s_4 = 2([x_1, x_2] \circ [x_3, x_4] - [x_1, x_3] \circ [x_2, x_4] + [x_1, x_4] \circ [x_2, x_3]).$$

If all x_i are symmetric variables then $[x_i, x_j]$ are skew symmetric and as such they commute in R. Therefore the standard polynomial s_4 equals in R twice the polynomial from the Identity (2). Thus the identities of $sl_2(K)$ above are satisfied in R provided all variables are symmetric.

Since all identities for $M_2(K)$ are satisfied as $*$-identities in R, one immediately obtains that when w_1 and w_2 are skew symmetric commutators of degrees ≥ 2 then $w_1(w_2L(a,b)) = (w_1L(a,b))w_2$ in R; see for example [12]. Thus we obtain the following lemma.

LEMMA 2.8: *The transformation L is well defined linear operator on $F_1 \oplus F_2 \oplus F_3$.*

Note that one may use the proof of [17, Lemma 1.2] in order to obtain once more that L is well defined.

PROPOSITION 2.9: *Assume that $f_i \in F_i$, $\deg f_i \geq 4$. Then in R, f_i can be written as follows.*

1. $f_1 = \sum_i \alpha_i [x_{i_1}, x_{i_2}][x_{i_3}, x_{i_4}] \prod_j L(a_{ij}, b_{ij})$. *Here* $\alpha_i \in K$, $i_1 < i_2$, $i_3 < i_4$, $i_1 \leq i_3$, $i_2 \leq i_4$.
2. $f_2 = \sum_i \beta_i [x_{i_1}, x_{i_2}] \prod_j L(a_{ij}, b_{ij})$, $\beta_i \in K$ *and* $i_1 < i_2$.
3. $f_3 = \sum_i \gamma_i [x_{i_1}, x_{i_2}, x_{i_3}] \prod_j L(a_{ij}, b_{ij})$, $\gamma_i \in K$ *and* $i_1 < i_2 \leq i_3$.

In all cases a_{ij} and b_{ij} are symmetric variables.

Proof: We need only prove the statement about the inequalities of the indices in (1). But it follows from the $*$-identity (2). ∎

We give a brief motivation for the steps that follow. We consider $F_1 \oplus F_2 \oplus F_3$ as a module over the commutative algebra generated by the operators L. We will find elements that span the F_i's. Subsequently, we will show that these elements when evaluated on the generic matrices with involution are linearly independent. Hence the algebra R will be isomorphic to the relatively free $*$-algebra determined by $(M_2(K), *)$.

The polynomials we consider are $*$-proper. Hence when computing such a polynomial on $M_2(K)$ we may dispense with the "scalar" component of the respective matrices. In other words, if $m_i \in M_2(K)$ and $m_i' = m_i - (1/2)\operatorname{tr} m_i$ then $f(m_1, \ldots, m_n) = f(m_1', \ldots, m_n')$ for every $*$-proper polynomial $f(x_1, \ldots, x_n)$ depending on symmetric variables only. Here we identify the field K with the centre of $M_2(K)$. Thus, any $*$-proper polynomial that depends on symmetric variables is $*$-identity for $M_2(K)$ if and only if it vanishes under substitutions by symmetric traceless matrices.

In the vector space $sl_2(K)$ one has the inner product $\langle a, b\rangle = a \circ b$ for a, $b \in sl_2(K)$. A standard argument shows that without loss of generality one may consider the base field K algebraically closed. Therefore if $i^2 = -1$ in K

we have the following basis of the vector space $M_2(K)$:

$$e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The matrices e_1, e_2, e_3 form a basis of $sl_2(K)$. Since we are interested in symmetric matrices then we may consider e_1 and e_2 only.

If we substitute x_1, x_2, a, b for traceless matrices we will get $[x_1, x_2]L(a, b) = [x_1, x_2] \circ (a \circ b)$. Here $a \circ b$ is the inner product in the span of e_1 and e_2. Now we employ the invariants of the orthogonal group for this inner product as described in [5], in a manner similar to that of [17], [10] and [12]. The invariants are described in terms of double tableaux. Let T be the double tableau

$$(12) \qquad \left(\begin{array}{ccccc|ccccc} p_{11} & p_{12} & \dots & \dots & p_{1m_1} & q_{11} & q_{12} & \dots & \dots & q_{1m_1} \\ p_{21} & p_{22} & \dots & \dots & p_{2m_2} & q_{21} & q_{22} & \dots & \dots & q_{2m_2} \\ \dots & \dots & \ddots & \dots & & \dots & \dots & \ddots & \dots & \\ p_{k1} & p_{k2} & \dots & p_{km_k} & & q_{k1} & q_{k2} & \dots & q_{km_k} & \end{array} \right)$$

where $m_1 \geq m_2 \geq \cdots \geq m_k$ and all p_{ij}, q_{ij} are integers.

When all entries of T are positive integers and $m_1 \geq 2$ then T is called 0-tableau. If $p_{11} = 0$ and all other entries of T are positive integers then T is called 1-tableau; when $m_1 \geq 2$ and $p_{11} = -1$, $p_{12} = 0$ while all other entries are positive integers, T is called 2-tableau.

We associate to T a polynomial $\varphi(T)$ in R, in the following way.

Definition 2.10: Let $T = (p_1, p_2, \dots, p_m \mid q_1, q_2, \dots, q_m)$ be a tableau consisting of one row.

1. If T is 0-tableau then we set $\varphi(T)$ to be

$$(1/4) \sum_{\sigma \in S_m} (-1)^\sigma [x_{p_1}, x_{p_2}][x_{q_{\sigma(1)}}, x_{q_{\sigma(2)}}]L(x_{p_3}, x_{q_{\sigma(3)}}) \dots L(x_{p_m}, x_{q_{\sigma(m)}}).$$

2. If T is 1-tableau then $\varphi(T)$ equals

$$-(1/4) \sum_{\sigma \in S_m} (-1)^\sigma [x_{q_{\sigma(1)}}, x_{q_{\sigma(2)}}, x_{p_2}]L(x_{p_3}, x_{q_{\sigma(3)}}) \dots L(x_{p_m}, x_{q_{\sigma(m)}}).$$

3. If T is 2-tableau with $m = 2$ then $\varphi(T) = -(1/4)[x_{q_1}, x_{q_2}]$. If otherwise $m \geq 3$ then we set

$$\varphi(T) = -(1/4) \sum_{\sigma \in S_m} (-1)^\sigma [x_{q_{\sigma(1)}}, x_{q_{\sigma(2)}}]L(x_{p_3}, x_{q_{\sigma(3)}}) \dots L(x_{p_m}, x_{q_{\sigma(m)}}).$$

In the general case, let $T_1, T_2, \ldots, T_k$ be the successive rows of T. Then we set $\varphi(T) = \varphi(T_1)l_2 \ldots l_k$ where we denote by l_j the product of (commuting) linear transformations

$$l_j = \sum_{\sigma \in S_m} (-1)^\sigma L(x_{p_{j1}}, x_{q_{j\sigma(1)}}) \ldots L(x_{p_{jm}}, x_{q_{j\sigma(m)}}), \quad m = m_j.$$

Here S_m stands for the symmetric group permuting $\{1, 2, \ldots, m\}$ and $(-1)^\sigma$ is the sign of σ.

Note that l_j is the "determinant" of the $m \times m$ matrix whose (a, b)-th entry is $L(x_{p_{ab}}, x_{q_{p\sigma(b)}})$.

LEMMA 2.11: *Suppose that T is k-tableau, $k = 0, 1, 2$. If $m_1 \geq 3$ then $\varphi(T) = 0$ in R.*

Proof: It is sufficient to prove the lemma in the case when T consists of one single row and $m_1 = 3$. If $T = (123 \mid 456)$ is 0-tableau then

$$\begin{aligned}\varphi(T) &= (1/8)[x_1, x_2]([x_4, x_5, x_3, x_6] - [x_4, x_6, x_3, x_5] + [x_5, x_6, x_3, x_4]) \\ &= (1/8)[x_1, x_2]([x_4, x_5, x_6, x_3] - [x_4, x_6, x_5, x_3] + [x_5, x_6, x_4, x_3]),\end{aligned}$$

since $[v, x_i, x_j] = [v, x_j, x_i]$ for any skew symmetric element v. But the second expression vanishes due to the Jacobi identity.

Now let $T = (012 \mid 345)$ be 1-tableau. Then $\varphi(T)$ equals

$$-(1/8)([x_3, x_4, x_2, x_5, x_1] - [x_3, x_5, x_2, x_4, x_1] + [x_4, x_5, x_2, x_3, x_1]) = 0$$

in the same manner as for 0-tableaux. Analogously, one deals with the case of a 2-tableau $T = (-101 \mid 234)$: $\varphi(T) = -(1/8)([x_2, x_3, x_1, x_4] - [x_2, x_4, x_1, x_3] + [x_3, x_4, x_1, x_2]) = 0$. ∎

Definition 2.12: The double tableau T is called (doubly) standard if $p_{ij} < p_{i,j+1}$, $q_{ij} < q_{i,j+1}$, $p_{ij} \leq q_{ij}$ and $q_{ij} \leq p_{i+1,j}$ for every i and j that make sense in the above inequalities.

Remark 2.13: Let V be the "generic" vector space with a basis consisting of the vectors $x_i = (x_{i1}, x_{i2}, \ldots, x_{in})$. Define an inner product by $x_i \circ x_j = \sum_k x_{ik} x_{jk}$. It is well known (see [5]) that the algebra A of invariants of the orthogonal group is generated by the products $x_i \circ x_j$ and it has a linear basis that is indexed by the standard tableaux T such that $m_1 \leq m$. If $T = (p_1 \ldots p_m \mid q_1 \ldots q_m)$ consists of one row then the corresponding basic element is $\tilde{T} = \det(x_{p_i} \circ x_{q_j})_{m\times m}$. If the rows of T are $T_1, T_2, \ldots, T_k$ then $\tilde{T} = \tilde{T_1}\tilde{T_2}\cdots\tilde{T_k}$.

Now we apply the above remark to the vector space spanned by the generic traceless symmetric matrices of order two. The inner product in it is defined by $u \circ v = (1/2)(uv + vu)$. Thus the polynomials $\tilde{T}$ such that T is standard and $m_1 \leq 2$ are linearly independent.

Definition 2.14: Denote by *Adm* the set of all double i-tableaux T, $i = 0, 1, 2$, such that T is standard and the length of the first row of T satisfies $m_1 \leq 2$.

PROPOSITION 2.15: *The polynomials $\{\varphi(T) \mid T \in Adm\}$ span the vector space $F_1 \oplus F_2 \oplus F_3$ of all $*$-proper polynomials in R that depend only on symmetric variables.*

Proof: We prove that every polynomial $f_k \in F_k$ is a linear combination of polynomials $\varphi(T_j)$ for some standard i-tableaux $T_j \in Adm$. Let T be the double tableau (12), we denote by $m(T) = m = (m_1, m_2, \ldots, m_k)$ the **shape** of T, and by $d = d(T)$ the sequence

$$(p_{11}, \ldots, p_{1,m_1}, q_{11}, \ldots, q_{1,m_1}, p_{21}, \ldots, q_{21}, \ldots, p_{k1}, \ldots, q_{k,m_k})$$

that is the **contents** of T. If T_1 and T_2 are two double tableaux then $T_1 < T_2$ if $m(T_1) < m(T_2)$ in the usual lexicographic order or else $m(T_1) = m(T_2)$ but $d(T_1) > d(T_2)$, again in the lexicographic order.

First let $f_1 \in F_1$; then according to Proposition 2.9 one writes $f_1 = \sum \alpha_i \varphi(T_i)$ where $\alpha_i \in K$ and T_i are 0-tableaux. Without loss of generality we may assume that all rows of T_i are of length ≤ 2. Thus it suffices to show that every $\varphi(T_i)$ can be expressed as a linear combination of polynomials corresponding to standard 0-tableaux. Set $T = T_i$.

Suppose that T is not standard. It is easy to see that we may consider that every half row of T is put in ascending order. If this is not so and the respective half row is not the first, then reordering its entries may result in a change of sign. If it is the first row then we use Proposition 2.9. Furthermore, if the violation of the standardness occurs below the first row, one applies verbatim the argument from [10, Lemma 2.7] and expresses $\varphi(T)$ as a combination of tableaux that are larger in the order defined above. Note that when the standardness breaks in one row of T then $\varphi(T)$ will be a combination of tableaux of the same shape but with smaller contents; if it breaks in two successive rows then the combination will involve in addition tableaux of larger shape.

If, on the other hand, we have $p_{1r} > q_{1r}$ then once again Proposition 2.9 resolves it. Therefore the only case to be considered is $q_{1r} > p_{2r}$, $r = 1$ or 2. But it is resolved using [12, Lemma 4.2, Proposition 4.2], and the remarks preceding Lemma 2.8. Therefore $\varphi(T)$ is a linear combination $\varphi(T) = \sum_i \beta_i \varphi(T_i)$, $\beta_i \in K$, $T_i > T$. In the sequel we consider the polynomials $\varphi(T_i)$ and the corresponding tableaux T_i. Treating them in the same manner we shall get tableaux that are still higher in the order. But this process must end up with standard tableaux. Note that it cannot be infinite since the polynomials under consideration have the same multidegree and hence we deal with a finite set of double tableaux.

Now let T be 2-tableau. As above we may suppose that T is not standard and that the violation of the standardness is of the type $q_{1r} > p_{2r}$. In this case one proceeds in the same manner as above using the corresponding result for the identities of the Lie algebra $sl_2(K)$ in [17, Proposition 2.1] instead of the arguments from [12].

The last case to consider is when T is 1-tableau, and as the former two cases, we suppose that the violation is $q_{1r} > p_{2r}$. It is dealt with using once again [17, Proposition 2.1], and we are done. ∎

THEOREM 2.16: *Let K be an infinite field of characteristic $p \neq 2$. The $*$-identities (1), (2) and (3) form a basis of the $*$-identities of $M_2(K)$ when $*$ is the transpose involution.*

Proof: Recall that I is the $*$-ideal generated by the identities (1), (2) and (3). We need to prove that $I = T(M_2(K), *)$. Since the inclusion $I \subseteq T(M_2(K), *)$ is immediate then we have the canonical epimorphism

$$R = K\langle X, Y\rangle / I \to K\langle X, Y\rangle / T(M_2(K), *) = \overline{R}.$$

Therefore the image of the set $\{\varphi(T) \mid T \in Adm\}$ spans the vector space of all $*$-proper polynomials in $\overline{R}$ that depend only on symmetric variables. The theorem will be proved if we show that the images of the polynomials $\varphi(T)$, $T \in Adm$, are linearly independent.

Here we make use of the following weak identities for the pair $(M_2(K), sl_2(K))$. First one has $[x_1, x_2]L(a, b) = [x_1, x_2] \circ (a \circ b)$. As a consequence of $[x_1 \circ x_2, x_3] = 0$ one obtains $[x_1, x_2] \circ x_3 = x_1 \circ [x_2, x_3]$. Then it is easy to verify that the weak identity

$$[x_1, x_2] \circ [x_3, x_4] = -4((x_1 \circ x_3)(x_2 \circ x_4) - (x_1 \circ x_4)(x_2 \circ x_3)) = -4\tilde{T}$$

holds for $T = (12 \mid 34)$; see for example [10, Lemma 2.1]. We will need the following lemma.

LEMMA 2.17: *Let T be i-tableau, $i = 0, 1, 2$. Then the following equalities hold in $\overline{R}$.*

1. *When T is 0-tableau, $\varphi(T) = -2\tilde{T}$.*
2. *When T is 1-tableau, $x_0 \circ \varphi(T) = -2\tilde{T}$.*
3. *When T is 2-tableau, $[x_{-1}, x_0] \circ \varphi(T) = \tilde{T}$.*

Here we use the same notation $\varphi(T)$ for the corresponding polynomials in R and for their images in $\overline{R}$.

Proof: The first statement was proved above. For the second, one observes that $x_0 \circ [x_3, x_2, x_1] = [x_1, x_2] \circ [x_3, x_4]$ is a weak identity for $(M_2(K), sl_2(K))$, according to [10, p. 616]. The last statement is proved in the same way as the first was. ∎

Now we return to our theorem. The polynomials $\tilde{T}$ for T standard and $m_1 \leq 2$ are linearly independent in $\overline{R}$. By Lemma 2.17 we have that the polynomials $\{\varphi(T) \mid T \in Adm\}$ are linearly independent in $\overline{R}$. Notice that for 0-tableaux the polynomial $\varphi(T)$ represents scalar matrices in $M_2(K)$; when T is 1-tableau then $\varphi(T)$ evaluates on symmetric traceless matrices, and for 2-tableaux, $\varphi(T)$ goes to skew symmetric matrices in $M_2(K)$.

Therefore $\{\varphi(T) \mid T \in Adm\}$ are linearly independent already in R, and they form a basis of the vector space $F_1 \oplus F_2 \oplus F_3$. Thus the theorem is proved. ∎

3. The symplectic involution

In this section $*$ is the symplectic involution on $M_2(K)$, that is

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^* = \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}.$$

Then it is immediate that the following are $*$-identities for $(M_2(K), *)$:

$$(13) \qquad [x_1, x_2] = 0, \quad [x, y] = 0$$

since the symmetric elements of $(M_2(K), *)$ are the scalar matrices.

As in the previous sections, it can be shown that the $*$-identities for $(M_2(K),*)$ are determined by the multihomogeneous ones.

THEOREM 3.1: *Let K be an infinite field of characteristic $p \neq 2$, and let $*$ be the symplectic involution on $M_2(K)$. All $*$-identities of this algebra follow from the identities (13).*

Proof: Let $f(x_1, \dots, x_m, y_1, \dots, y_n)$ be $*$-identity for $M_2(K)$. Since x_i are central then

$$f(x_1, \dots, x_m, y_1, \dots, y_n) = x_1^{k_1} \cdots x_m^{k_m} g(y_1, \dots, y_n)$$

for some polynomial g that depends only on skew symmetric variables. Obviously g is again $*$-identity for $M_2(K)$. On the other hand, the skew symmetric elements of $(M_2(K), *)$ coincide with $sl_2(K)$, therefore g is weak identity for the pair $(M_2(K), sl_2(K))$. But then according to [10, Theorem 3.2] we may write g in the following way:

$$g(y_1, y_2, \dots, y_n) = \sum_i \alpha_i u_i [v_i \circ w_i, t_i] r_i$$

where $\alpha_i \in K$, u_i, t_i and r_i are associative polynomials and v_i and w_i are commutators of degree ≥ 2, all depending only on skew symmetric variables. Since $[y_1, y_2]^* = -[y_1, y_2]$ then v_i and w_i are skew symmetric elements. On the other hand, $y_1 \circ y_2 = (1/2)(y_1 y_2 + y_2 y_1) = (1/2)(y_1 y_2 + (y_1 y_2)^*)$ is symmetric element. Therefore $[v_i \circ w_i, t_i] = 0$ follows from (13), and the theorem is proved. ∎

Remark 3.2: It would be interesting to know the identities with involution for other matrix algebras. On the other hand, it should be rather difficult. A first step in this direction could be the following.

Describe the basis of the $*$-identities for $M_4(K)$ over an infinite field (or even in characteristic 0) where $*$ stands for the symplectic involution.

Very probably, the description of the minimal $*$-identities given by D'Amour and Racine in [3], [4] would help in obtaining such a basis.

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